NORMAL POSITIVE LINEAR SYSTEMS AND ELECTRICAL CIRCUITS

Summary. The notion of normal positive electrical circuits is introduced and some their specific properties are investigated. New state matrices of positive linear systems and electrical circuits are proposed and their properties are analyzed. It is shown that positive electrical circuits with diagonal state matrices are normal for all values of resistances, inductances and capacitances.

Keywords: normal, positive, linear, system, electrical circuit

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs [2, 12]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The notions of controllability and observability have been introduced by Kalman in [25, 26] and they are the basic concepts of the modern control theory [1, 4, 5, 8, 9, 17, 24, 28]. The controllability, reachability and observability of linear systems and electrical circuits have been investigated in [6, 7, 13, 15, 16, 27]. The asymptotic stability of positive standard and fractional linear systems has been addressed in [3, 12, 23].
The specific duality and stability of positive electrical circuits have been analyzed in [18] and positive systems and electrical circuits with inverse state matrices in [14]. The stability of continuous-time and discrete-time linear systems with inverse state matrices has been investigated in [22]. The reduction of linear electrical circuits with complex eigenvalues to linear electrical circuits with real eigenvalues has been considered in [21].

Standard and positive electrical circuits with zero transfer matrices have been investigated in [19] and the normal positive electrical circuits have been introduced in [10]. In this paper the normal positive linear systems and electrical circuits are investigated.

The paper is organized as follows. In section 2 some preliminaries concerning positive linear continuous-time systems are recalled. Some properties of the transfer matrices of positive linear systems are presented in section 3. Normal positive linear systems are analyzed in section 4. Normal positive linear electrical circuits are introduced and investigated in section 5. Concluding remarks are given in section 6.

The following notation will be used: \( \mathbb{R} \) - the set of real numbers, \( \mathbb{R}^{n \times m} \) - the set of \( n \times m \) real matrices, \( \mathbb{R}^{n \times m}_+ \) - the set of \( n \times m \) real matrices with nonnegative entries and \( \mathbb{R}^n_+ = \mathbb{R}^{n \times 1}_+ \), \( M_n \) - the set of \( n \times n \) Metzler matrices (real matrices with nonnegative off-diagonal entries), \( I_n \) - the \( n \times n \) identity matrix.

2. PRELIMINARIES

Consider the continuous-time linear system
\[
\dot{x} = Ax + Bu,
\]
\[
y = Cx,
\]
where \( x = x(t) \in \mathbb{R}^n \), \( u = u(t) \in \mathbb{R}^m \), \( y = y(t) \in \mathbb{R}^p \) are the state, input and output vectors and \( A \in \mathbb{R}^{n \times n}_+ \), \( B \in \mathbb{R}^{n \times m}_+ \), \( C \in \mathbb{R}^{p \times n}_+ \).

**Definition 2.1.** [12] The linear system (2.1) is called (internally) positive if \( x(t) \in \mathbb{R}_+^n \) and \( y(t) \in \mathbb{R}_+^p \), \( t \geq 0 \) for any initial conditions \( x_0 \in \mathbb{R}_+^n \) and all inputs \( u(t) \in \mathbb{R}_+^m \), \( t \geq 0 \).

**Theorem 2.1.** [12] The linear system (2.1) is positive if and only if
\[
A \in M_n, \quad B \in \mathbb{R}^{n \times m}_+, \quad C \in \mathbb{R}^{p \times n}_+.
\]

**Definition 2.2.** [12] The positive linear system (2.1) for \( u(t) = 0 \) is called asymptotically stable if
\[
\lim_{t \to \infty} x(t) = 0 \quad \text{for all} \quad x_0 \in \mathbb{R}_+^n.
\]

**Theorem 2.2.** [12] The positive linear system (2.1) for \( u(t) = 0 \) is asymptotically stable (the matrix \( A \) is Hurwitz) if and only if all coefficients of the characteristic polynomial


\[ p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \]  

are positive, i.e. \( a_k > 0 \) for \( k = 0,1,\ldots,n-1 \).

We shall consider the positive system (2.1) with the matrix \( A \) of the form

\[
A_i = \begin{bmatrix}
-s_1 & a_1 & 0 & \cdots & 0 & 0 \\
0 & -s_2 & a_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -s_{n-1} & a_{n-1} \\
0 & 0 & 0 & \cdots & 0 & -s_n
\end{bmatrix} \quad \text{or} \quad A_2 = \begin{bmatrix}
a_1 & -s_2 & 0 & \cdots & 0 & 0 \\
a_2 & -s_3 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -s_{n-1} & a_{n-1} \\
0 & 0 & 0 & \cdots & 0 & -s_n
\end{bmatrix},
\tag{2.5}
\]

\( a_k > 0, \ k = 1,\ldots,n-1 \).

**Theorem 2.3.** [12] The positive system with (2.5) is asymptotically stable if and only if

\[
\text{Re} s_k < 0 \quad \text{for} \quad k = 1,\ldots,n.
\tag{2.6}
\]

**Definition 2.3.** [12] The positive system (2.1) is called reachable in time \([0,t_f]\) if for any given final state \( x_f \in \mathbb{R}^n_+ \) there exists an input \( u(t) \in \mathbb{R}^m_+ \) for \( t \in [0,t_f] \) that steers the state \( x(t) \) from zero initial state \( x(0) = 0 \) to the final state \( x_f \), i.e. \( x(t_f) = x_f \).

**Definition 2.4.** [12] A real matrix \( A \in \mathbb{R}^{n \times n}_+ \) is called monomial if each its row (column) contains only one positive entry and the remaining entries are zero.

**Theorem 2.4.** [12] The positive system (2.1) is reachable if the matrix

\[
R_f = \int_0^{t_f} e^{A \tau} BB^T e^{A^T \tau} d\tau, \quad t_f > 0
\tag{2.7}
\]

is monomial.

The input

\[
u(t) = B^T e^{A^T(t-t) - \tau} R_f^{-1} x_f \in \mathbb{R}^{m \times 1}_+, \quad t \in [0,t_f]
\tag{2.8}
\]

steers the state \( x(t) \) of the system from \( x(0) = 0 \) to \( x(t_f) = x_f \).

The positive system (2.1) is reachable in time \([0,t_f]\) if and only if \( A \in M_n \) is diagonal and \( B \in \mathbb{R}^{m \times n}_+ \) has \( m \) linearly independent monomial columns.

**Definition 2.5.** [12] The positive system (2.1) is called observable in time \([0,t_f]\) if knowing the output \( y(t) \in \mathbb{R}^p_+ \) and the input \( u(t) \in \mathbb{R}^m_+ \) it is possible to find the unique initial condition \( x(0) \in \mathbb{R}^n_+ \).

**Theorem 2.5.** [12] The positive system (2.1) is observable in time \([0,t_f]\) if the matrix

\[
O_f = \int_0^{t_f} e^{A \tau} Ce^{A^T \tau} C^T d\tau, \quad t_f > 0
\tag{2.9}
\]

is monomial.
3. TRANSFER MATRICES OF POSITIVE LINEAR SYSTEMS

The transfer matrix of the positive linear system (2.1) is given by

\[ T(s) = C[I_n s - A]^{-1} B \in \mathbb{R}^{p \times m}(s), \quad (3.1) \]

where \( \mathbb{R}^{p \times m}(s) \) is the set of \( p \times m \) rational matrices in \( s \).

**Theorem 3.1.** If the matrix \( A \in M_n \) given by (2.5) is asymptotically stable (Hurwitz) and \( B \in \mathbb{R}^{p \times m}_+ \), \( C \in \mathbb{R}^{p \times n}_+ \) then all coefficients of the transfer matrices

\[ T_1(s) = C[I_n s - A_1]^{-1} B, \quad T_2(s) = C[I_n s - A_2]^{-1} B \quad (3.2) \]

are nonnegative.

**Proof.** If \( A_k \) is Hurwitz and \( a_{ki} > 0 \), \( k = 1, ..., n - 1 \) then the entries of the inverse matrix

\[ [I_n s - A_1]^{-1} = \begin{bmatrix} \frac{1}{s + s_1} & a_{12} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & \frac{1}{s + s_2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s + s_{n-1}} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & \frac{1}{s + s_n} \end{bmatrix} \]

are rational functions with nonnegative coefficients.

Therefore, if \( B \in \mathbb{R}^{p \times m}_+ \) and \( C \in \mathbb{R}^{p \times n}_+ \) then all coefficients of the transfer matrix \( T_i(s) \) are nonnegative.

The proof for \( T_2(s) \) is similar (dual). \( \square \)

**Example 3.1.** Consider the transfer function of the positive system (3.1) with

\[ A = A_k = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}. \quad (3.4) \]

In this case using (3.1) and (3.4) we obtain
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\[ T_1(s) = C[I_3 s - A_1]^{-1} B = [1 \quad 3 \quad 2] \begin{bmatrix} s + 1 & -2 & 0 \\ 0 & s + 2 & -1 \\ 0 & 0 & s + 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ = \frac{1}{(s + 1)(s + 2)(s + 3)} \begin{bmatrix} 1 \quad 3 \quad 2 \end{bmatrix} \begin{bmatrix} (s + 2)(s + 3) & 2(s + 3) & 2 \\ 0 & (s + 1)(s + 3) & s + 1 \\ 0 & 0 & (s + 1)(s + 2) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ = \frac{2s^2 + 9s + 9}{s^3 + 6s^2 + 11s + 6}. \]

The transfer function is minimal-phase since its zeros \( z_1 = -1.5, \ z_2 = -3 \) are negative. After cancellation of the zero \( z_2 = -3 \) with the pole \( \lambda_5 = -3 \) we obtain

\[ T_1(s) = \frac{2s + 3}{s^2 + 3s + 2}. \] (3.6)

It is easy to check that if

\[ A_i = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \ B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ C_i = [1 \ 2 \ 1] \] (3.7)

then

\[ T_1(s) = C_i[I_3 s - A_i]^{-1} B_i = [1 \quad 2 \quad 1] \begin{bmatrix} s + 1 & -2 & 0 \\ 0 & s + 2 & -1 \\ 0 & 0 & s + 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{s + 1}. \] (3.8)

In this case we have

\[ \text{rank} [B_i \ A_i B_i \ A_i^2 B_i] = \text{rank} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -5 \\ 1 & -3 & 9 \end{bmatrix} = 3 = n \] (3.9)

and

\[ \text{rank} \begin{bmatrix} C_i \\ C_i A_i \\ C_i A_i^2 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix} = 1 < n = 3. \] (3.10)

Therefore, the standard pair \((A_i, B_i)\) is controllable, but the pair \((A_i, C_i)\) is unobservable.

Consider the SISO (single-input \((m = 1)\) single-output \((p = 1)\)) positive linear system with \(A_i\) given by (2.5) and

\[ B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n_+, \ C_i \in \mathbb{R}^{1\times n}_+. \] (3.11)
It is easy to check that
\[
\text{rank}[B_1 \ A_1 B_1 \cdots A_1^{n-1}B_1] = n \text{ if } a_k > 0, \ k = 1, \ldots, n-1. \tag{3.12}
\]

Let \(z_1, z_2, \ldots, z_{n-1}\) be the zeros (the roots of \(n(s) = 0\)) and \(p_1, p_2, \ldots, p_n\) the poles (the roots of \(d(s) = 0\)) of the transfer function
\[
T_1(s) = C_1[I_n s - A_1]^{-1} B_1 = \frac{n(s)}{d(s)}. \tag{3.13}
\]

**Theorem 3.2.** If
\[
\text{rank} \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix} < n \tag{3.14}
\]
then at least one zero of (3.13) is equal to its poles.

**Proof.** It is well-known that if (3.14) holds then the zeros and poles cancellation occurs in (3.13). It happens only if at least one zero of (3.13) is equal to its poles. \(\square\)

Now let us consider the SISO positive system with \(A_2\) given by (2.5) and
\[
B_2 \in \mathbb{R}_+^n, \ C_2 = [0 \ \cdots \ 0 \ 1] \in \mathbb{R}_+^{1 \times n}. \tag{3.15}
\]

It is easy to check that
\[
\text{rank} \begin{bmatrix} C_2 \\ C_2 A_2 \\ \vdots \\ C_2 A_2^{n-1} \end{bmatrix} = n \text{ if } a_k > 0, \ k = 1, \ldots, n-1. \tag{3.16}
\]

**Theorem 3.3.** Let \(p_1, p_2, \ldots, p_n\) be the poles and \(z_1, z_2, \ldots, z_{n-1}\) the zeros of the transfer function
\[
T_2(s) = C_2[I_n s - A_2]^{-1} B_2. \tag{3.17}
\]

If
\[
\text{rank}[B_2 \ A_2 B_2 \cdots A_2^{n-1}B_2] < n \tag{3.18}
\]
then at least one zero of (3.17) is equal to its poles.

**Proof.** The proof is dual to the proof of Theorem 3.2.

### 4. NORMAL POSITIVE LINEAR SYSTEMS

Consider the transfer matrix of the form
\[
T(s) = \frac{N(s)}{d(s)} \in \mathbb{R}^{p \times m}(s), \tag{4.1a}
\]
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where \( N(s) \in \mathbb{R}^{p \times m}[s] \) is the polynomial matrix and \( d(s) \) is the least common denominator of the form

\[
d(s) = s^n + a_{n-1}s^{n-1} + ... + a_1s + a_0.
\]

(4.1b)

**Definition 4.1.** The positive linear system with (4.1) is called normal if every nonzero second order minor of \( N(s) \) is divisible (with zero remainder) by the polynomial \( d(s) \).

The normal systems are insensitive to the change of their parameters [11].

**Definition 4.2.** The state matrix \( A \) of the linear system (2.1) is called cyclic if its minimal polynomial \( \varphi(s) \) is equal to its characteristic polynomial

\[
\varphi(s) = \det[I_n s - A].
\]

(4.2)

The minimal polynomial \( \psi(s) \) is related to its characteristic polynomial \( \varphi(s) \) by [11]

\[
\psi(s) = \frac{\varphi(s)}{D_{n-1}(s)},
\]

(4.3)

where \( D_{n-1}(s) \) is the greatest common divisor of all \( n-1 \) order minors of the matrix \([I_n s - A]\).

Therefore, \( \psi(s) = \varphi(s) \) if and only if \( D_{n-1}(s) = 1 \).

**Theorem 4.1.** The matrices \( A_1 \) and \( A_2 \) defined by (2.5) are cyclic.

**Proof.** By Definition 4.2 and (4.3) the matrices \( A_1 \) and \( A_2 \) are cyclic if and only if the greatest common divisor of all \( n-1 \) order minors of the matrices

\[
[I_n s - A_1] = \begin{bmatrix}
s + s_1 & -a_1 & 0 & \cdots & 0 & 0 \\
0 & s + s_2 & -a_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & s + s_{n-1} & -a_{n-1} \\
0 & 0 & 0 & \cdots & 0 & s + s_n \end{bmatrix},
\]

(4.4)

\[
[I_n s - A_2] = \begin{bmatrix}
s + s_1 & 0 & 0 & \cdots & 0 & 0 \\
-a_1 & s + s_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & s + s_{n-1} & 0 \\
0 & 0 & 0 & \cdots & -a_{n-1} & s + s_n \end{bmatrix}
\]

are \( D_{n-1}(s) = 1 \). It is easy to see that the minors corresponding to the first column and the \( n \)-th row of the matrix \([I_n s - A_1]\) and to the first row and the \( n \)-th column of the matrix \([I_n s - A_2]\) are equal to \( a_1a_2...a_{n-1} \). Therefore, \( D_{n-1}(s) = 1 \) and the matrices \( A_1 \) and \( A_2 \) are cyclic. \( \square \)

**Theorem 4.2.** The positive linear system with the matrices \( A_1 \) and \( A_2 \) defined by (2.5) is normal for any \( B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{m \times n} \).
Proof. By Definition 4.1 the positive linear system with \(A_1, A_2\) defined by (2.5) and any \(B \in \mathbb{R}_+^{nxm}, C \in \mathbb{R}_+^{pxn}\) is normal if every nonzero second order minor of the matrix \(N(s) = C[I_s - A_1]_ad B\) is divisible by the polynomial \(\det[I_s - A_1]\).

Let \(Z_{j_1j_2...j_q}\) be the minor of the matrix \(Z\) with its \(i_1, i_2, \ldots, i_q\) rows and \(j_1, j_2, \ldots, j_q\) its columns. Then it is well-known \([20]\) that the \(q\)-minor of the matrix \(Z = PQ\) is given by

\[
Z_{j_1j_2...j_q} = \sum_{1 \leq k_1 < \cdots < k_q} P_{ik_1}Q_{jk_1}^{k_2...k_q}.
\]  

(4.5)

Note that the minors of the matrices \(B\) and \(C\) are independent of \(s\). Using (4.5) for the matrix \(C[I_s - A_1]_ad B\) it is easy to see that its every nonzero second order minor is divisible by \(\det[I_s - A_1]\) since by Theorem 4.1 the matrix \(A_1, A_2\) is cyclic. Therefore, the positive linear system with \(A_1, A_2\) and any \(B \in \mathbb{R}_+^{nxm}, C \in \mathbb{R}_+^{pxn}\) is normal. \(\square\)

Example 4.1. Consider the positive linear system with the matrices

\[
A_1 = \begin{bmatrix}
-1 & a_1 & 0 \\
0 & -2 & a_2 \\
0 & 0 & -3 \\
\end{bmatrix}, \quad a_k > 0 \text{ for } k = 1, 2, \quad B = \begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
b_{31} & b_{32} \\
\end{bmatrix} \in \mathbb{R}_+^{3 \times 2},
\]

(4.6)

\[
C = \begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
\end{bmatrix} \in \mathbb{R}_+^{2 \times 3}.
\]

Taking into account that

\[
d(s) = \det[I_s - A_1] = \begin{vmatrix}
s + 1 & -a_1 & 0 \\
0 & s + 2 & -a_2 \\
0 & 0 & s + 3 \\
\end{vmatrix} = (s + 1)(s + 2)(s + 3) = s^3 + 6s^2 + 11s + 6
\]

(4.7)

and

\[
[I_s - A_1]_ad = \begin{bmatrix}
(s + 2)(s + 3) & a_1(s + 3) & a_1a_2 \\
0 & (s + 1)(s + 3) & a_2(s + 1) \\
0 & 0 & (s + 1)(s + 2) \\
\end{bmatrix}
\]

(4.8)

we obtain

\[
N(s) = C[I_s - A_1]_ad B_1
\]

\[
= \begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
0 & (s + 1)(s + 2) & a_2(s + 1) \\
0 & 0 & (s + 1)(s + 2) \\
\end{bmatrix}
\]

(4.9a)

\[
= \begin{bmatrix}
n_{11}(s) & n_{12}(s) \\
n_{21}(s) & n_{22}(s) \\
\end{bmatrix},
\]
where

\[ n_{11}(s) = b_{21}b_{31}a_2c_{12}(s+1) + b_{21}a_1c_{11}(s+3) + b_{21}b_{31}c_{13}(s+1)(s+2) + b_{21}b_{11}c_{11}(s+2)(s+3) \]
\[ + b_{22}c_{13}(s+1)(s+3) + b_{22}b_{32}a_2c_{11}, \]

\[ n_{12}(s) = b_{22}b_{32}a_2c_{12}(s+1) + b_{22}a_1c_{11}(s+3) + b_{22}b_{32}c_{13}(s+1)(s+2) + b_{22}b_{12}c_{11}(s+2)(s+3) \]
\[ + b_{22}c_{13}(s+1)(s+3) + b_{22}b_{32}a_2c_{11}, \]

\[ n_{21}(s) = b_{21}b_{31}a_2c_{22}(s+1) + b_{21}a_1c_{21}(s+3) + b_{21}b_{31}c_{23}(s+1)(s+2) + b_{21}b_{12}c_{21}(s+2)(s+3) \]
\[ + b_{21}c_{23}(s+1)(s+3) + b_{21}b_{31}a_2c_{21}, \]

\[ n_{22}(s) = b_{22}b_{32}a_2c_{22}(s+1) + b_{22}a_1c_{21}(s+3) + b_{22}b_{32}c_{23}(s+1)(s+2) + b_{22}b_{12}c_{21}(s+2)(s+3) \]
\[ + b_{22}c_{23}(s+1)(s+3) + b_{22}b_{32}a_2c_{21}. \]

Therefore, the positive linear system with (4.6) is normal.

Note that the matrices (2.5) for \( a_k = 0, \ k = 1, \ldots, n-1 \) are equal and have the diagonal form

\[ A_d = \text{diag}[-s_1, -s_2, \ldots, -s_n]. \] (4.10)

In this particular case Theorem 4.2 has the following form.

**Theorem 4.3.** The positive linear system with (4.10) and any \( B \in \mathbb{R}^{m \times m}_+, \ C \in \mathbb{R}^{k \times n}_+ \) is normal.

### 5. NORMAL POSITIVE LINEAR ELECTRICAL CIRCUITS

Consider linear electrical circuits composed of resistors, capacitors, coils and voltage (current) sources. As the state variables (the components of the state vector \( x(t) \)) we choose the voltages on the capacitors and the currents in the coils. Using Kirchhoff’s laws we may describe the linear circuits in transient states by the state equations

\[ \dot{x} = Ax + Bu, \] (5.1a)
\[ y = Cx, \] (5.1b)

where \( x = x(t) \in \mathbb{R}^n, \ u = u(t) \in \mathbb{R}^m, \ y = y(t) \in \mathbb{R}^p \) are the state, input and output vectors and \( A \in \mathbb{R}^{n \times n}_+, \ B \in \mathbb{R}^{m \times n}_+, \ C \in \mathbb{R}^{p \times n}_+. \)

**Definition 5.1.** [23] The linear electrical circuit (5.1) is called (internally) positive if the state vector \( x(t) \in \mathbb{R}^n_+ \) and output vector \( y(t) \in \mathbb{R}^p_+, \ t \geq 0 \) for any initial conditions \( x_0 \in \mathbb{R}^n_+ \) and all inputs \( u(t) \in \mathbb{R}^m_+, \ t \geq 0. \)

**Theorem 5.1.** [23] The linear electrical circuit (5.1) is positive if and only if

\[ A \in M_n_+, \ B \in \mathbb{R}^{m \times n}_+, \ C \in \mathbb{R}^{p \times n}_+. \] (5.2)

The transfer matrix of the linear electrical circuit described by (5.1) can be always written in the form (4.1a).

**Definition 5.2.** The positive linear electrical circuit is called normal if every nonzero second order minor of \( N(s) \) is divisible by \( d(s) \).
Example 5.1. Consider the linear electrical circuit shown on Fig. 1 with given resistances $R_k$, inductances $L_k$, $k=1,2,3$ and source voltages $e_1, e_2$.

![Electrical Circuit](image)

Fig. 1. Electrical circuit of Example 5.1

Using the mesh method for the electrical circuit we obtain

$$
\begin{bmatrix}
L_{11} & -L_{12} \\
-L_{21} & L_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{di_1}{dt} \\
\frac{di_2}{dt}
\end{bmatrix}
=\begin{bmatrix}
-R_{11} & R_{12} \\
R_{21} & -R_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
+\begin{bmatrix}e_1 \\
e_2\end{bmatrix},
$$

(5.3a)

where $i_1 = i_1(t)$, $i_2 = i_2(t)$ are the mesh currents and

$$
R_{11} = R_1 + R_3, \quad R_{12} = R_1 = R_2, \quad R_{22} = R_2 + R_3, \quad L_{11} = L_1 + L_3, \\
L_{42} = L_2 + L_3, \quad L_{22} = L_2 + L_3.
$$

(5.3b)

The inverse matrix

$$
L^{-1} = \begin{bmatrix}
L_{11} & -L_{12} \\
-L_{21} & L_{22}
\end{bmatrix}^{-1} = \frac{1}{L_1(L_2 + L_3) + L_2L_3}
\begin{bmatrix}
L_{22} & L_{42} \\
L_{21} & L_{11}
\end{bmatrix}
$$

(5.4)

has all positive entries.

From (5.3a) we obtain

$$
\frac{d}{dt}\begin{bmatrix}i_1 \\
i_2\end{bmatrix}
= A\begin{bmatrix}i_1 \\
i_2\end{bmatrix} + B\begin{bmatrix}e_1 \\
e_2\end{bmatrix},
$$

(5.5a)

where

$$
A = L^{-1}\begin{bmatrix}
-R_{11} & R_{12} \\
R_{21} & -R_{22}
\end{bmatrix}
= \frac{1}{L_1(L_2 + L_3) + L_2L_3}
\begin{bmatrix}
-L_2(R_1 + R_3) + L_3R_3 - L_3R_2 \\
L_4R_3 - L_3R_1 & -L_4(R_2 + R_3) + L_3R_2
\end{bmatrix},
$$

(5.5b)

$$
B = L^{-1} \in \mathbb{R}^{2 \times 2}.
$$

Note that if

$$
L_4R_3 = L_3R_1 \text{ and } L_2R_3 - L_3R_2 > 0
$$

(5.6a)
then the matrix $A$ has the form of the matrix $A_1$ defined by (2.5) and for
\[ L_2 R_3 = L_2 R_2 \] (5.6b)
the form of the matrix $A_2$. In both cases the electrical circuit is positive.

These considerations can be easily extended to $n$-mesh linear electrical circuits.

Following [23] let us consider the linear electrical circuit shown in Fig. 2 with given resistances $R_k$, $k = 1, \ldots, 8$, inductances $L_2$, $L_4$, $L_6$, $L_8$, capacitances $C_1$, $C_3$, $C_5$, $C_7$ and source voltages $e_0$, $e_2$, $e_4$, $e_6$, $e_8$.

![Positive electrical circuit](image)

**Fig. 2. Positive electrical circuit**

**Rys. 2. Dodatni obwód elektryczny**

Using Kirchhoff’s laws we may write the equations
\[ e_0 = u_k + R_k C_k \frac{du_k}{dt}, \quad k = 1, 3, 5, 7, \] (5.7a)
\[ e_0 + e_j = R_j i_j + L_j \frac{di_j}{dt}, \quad j = 2, 4, 6, 8. \] (5.7b)

The equations can be written in the form
\[ \frac{d}{dt} \begin{bmatrix} u \\ i \end{bmatrix} = A \begin{bmatrix} u \\ i \end{bmatrix} + B e, \] (5.8a)
where
\[ u = \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ i_4 \\ i_6 \end{bmatrix}, \quad i = \begin{bmatrix} i_2 \\ i_4 \\ i_6 \end{bmatrix}, \quad e = \begin{bmatrix} e_0 \\ e_2 \\ e_4 \\ e_6 \end{bmatrix} \] (5.8b)
and
The matrix $A \in M_{8}$ is diagonal and asymptotically stable and $B \in \mathbb{R}_{+}^{8 \times 5}$. Therefore, the electrical circuit is positive for any values of the resistances, inductances and capacitances and from Theorem 4.3 we have the following important theorem.

**Theorem 5.2.** Positive linear electrical circuit with diagonal matrix $A \in M_{n}$ and $B \in \mathbb{R}_{+}^{m \times n}$, $C \in \mathbb{R}_{+}^{p \times n}$ is normal for any values of the resistances, inductances and capacitances.

### 6. CONCLUDING REMARKS

The notion of normal positive electrical circuit has been introduced and some specific properties of this class have been investigated. New state matrices of the positive linear systems and electrical circuits have been introduced and their properties have been analyzed (Theorems 3.2, 3.3, 4.1, 4.2 and 4.3). It has been shown that the positive electrical circuits with diagonal state matrices are normal for all values of their resistances, inductances and capacitances (Theorem 5.2). The considerations have been illustrated by numerical examples.

The considerations can be extended to fractional linear systems and electrical circuits.

**Acknowledgments**

The studies have been carried out in the framework of work No. S/WE/1/2016 and financed from the funds for science by the Polish Ministry of Science and Higher Education.

**REFERENCES**

Normal positive linear systems...


